MIDTERM: MMATH LINEAR ALGEBRA

Date: **13th September 2019**
The Total points is **108** and the maximum you can score is **100** points.

All vector spaces considered below are assumed to be finite dimensional.

(1) (7+7+7+7=28 points) Answer the following multiple choice questions about each of them. Write all correct options. No justification needed. **No partial credit will be given if a correct option is missing or an incorrect option is written.**

(a) Let A and B be two distinct bases of a vector space V. Which of the following statements are true?
  (i) $A \cup B$ is a generating set.
  (ii) $A \cup B$ is a linearly independent set.
  (iii) A and B have same number of elements.
  (iv) If $C \subset A \cup B$ then C is either a linearly independent set or a generating set.

(b) Let $\phi : V \to W$ be a linear map of vector spaces. Let A be the matrix of ϕ with respect to the bases $(\mathscr{B}, \mathscr{C})$ of V and W respectively. Which of the following are true?
  (i) $\operatorname{rank}(A) = \dim(\ker(\phi))$
  (ii) $\operatorname{rank}(A) = \dim(\operatorname{im}(\phi))$
  (iii) $\operatorname{rank}(A) + \dim(\ker(\phi)) = \dim(V)$
  (iv) $\operatorname{rank}(A) + \dim(\operatorname{im}(\phi)) = \dim(W)$

(c) Let V and W be vector spaces. Let ϕ and ψ be linear operators V and W respectively. Consider the linear operator θ on $V \oplus W$ given by $\theta(v, w) = (\phi(v), \psi(w))$ for $v \in V$ and $w \in W$. Which of the following are true?
  (i) $\det(\theta) = \det(\phi)\det(\psi)$
  (ii) $\operatorname{tr}(\theta) = \operatorname{tr}(\phi) + \operatorname{tr}(\psi)$
  (iii) The minimal polynomial of θ is the product of the minimal polynomials of ϕ and ψ.
  (iv) The characteristic polynomial of θ is the product of the characteristic polynomials of ϕ and ψ.

(d) Let A and B be two hermitian matrices. Which of the following statements are true?
  (i) AB is hermitian.
  (ii) A^2 is hermitian.
  (iii) $A + B$ is hermitian.
  (iv) e^A is hermitian.

(2) (8+8+8+8+8=40 points) Prove or disprove (using a counterexample) the following statements.
  (a) Let A be a square matrix of rank 1. Then $A = xy^T$ for some coloumn vectors x and y.
  (b) Every real square matrix is similar to a real upper-triangular matrix.
  (c) Every real symmetric matrix is similar to a real diagonal matrix.
  (d) Let $(V, \langle \cdot, \cdot \rangle)$ be a vector space together with a symmetric bilinear form. Let A be the matrix of the bilinear form with respect to some basis. If $\det(A) = 1$ then A is positive definite.
  (e) Let $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ be two vectors in $\mathbb{C}^2$. The function $\langle X, Y \rangle = x_1y_1 + ix_1y_2 - ix_2y_1 + x_2y_2$ from $\mathbb{C}^2$ to $\mathbb{C}$ is a hermitian form.

(3) (10 points) Let A be a real orthogonal matrix of determinant -1. Show that -1 is an eigenvalue of A.

(4) (8+10+12=30 points) Define hermitian space. Let T be a linear operator on a hermitian space V. Define adjoint of T. When is a linear operator called normal? Show that $T - T^*$ is diagonalizable. Also show that if the dimension of V as a $\mathbb{C}$-vector space is even then the determinant of $T - T^*$ is a real number.